Koopman Operator and Control

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Outline

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Introduction to Koopman Operator

Given an autonomous dynamical system

$$x_{t+1} = f(x_t), \tag{1}$$

where $x_t \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \Omega$.

- f may be unknown or too complex for analysis.

We are interested in how the state x_t evolves.

Define a function (an observable) $g : \Omega \to \mathbb{R}$ in $\mathcal{H} := L^2(\Omega, \mu)$[1].

- Indirectly measures states, e.g., $g(x_t)$, $g(x_{t+1}) = g(f(x_t))$.
- A new observable $g' := g \circ f(x)$ indirectly monitors the evolution of f.

[1] g is typically in the Hilbert space $\mathcal{H}$.

Introduction to Koopman Operator

We define the Koopman operator $\mathcal{K}: \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{K}(g)(x) = g \circ f(x). \tag{2}$$

- $\mathcal{K}$ maps one function to another function.
- For a fixed g, $\mathcal{K}(g)(x)$ measures one-step state evolution if the current state is x.

$\mathcal{K}$ is an infinite-dimensional linear operator due to composition.

$$\begin{aligned}
\mathcal{K}(\alpha_1 g_1 + \alpha_2 g_2)(x) &= (\alpha_1 g_1 + \alpha_2 g_2) \circ f(x) \\
&= \alpha_1 g_1 \circ f(x) + \alpha_2 g_2 \circ f(x) \\
&= \alpha_1 \mathcal{K}(g_1)(x) + \alpha_2 \mathcal{K}(g_2)(x), \ \forall g_1, g_2 \in \mathcal{H}, \alpha_1, \alpha_2 \in \mathbb{R}.
\end{aligned}$$

Introduction to Koopman Operator

Question: How does $\mathcal{K}$ help?
Answer: Finite-dimensional representation of f.

Let $\boldsymbol{g} = [g_1, \ldots, g_m]$ be m observables. We have

$$\mathcal{K}(\boldsymbol{g}) = [g_1 \circ f, \ldots, g_m \circ f].$$

If $g_i \circ f \in \text{span}\{g_1, \ldots, g_m\}$ $\forall i$, i.e.,

$$\exists\ \alpha_{i1}, \ldots, \alpha_{im} \text{ s.t. } g_i \circ f = \alpha_{i1} g_1 + \cdots + \alpha_{im} g_m.$$

Then, we have

$$\boldsymbol{g}(x^+) = \boldsymbol{g}^+(x) := \mathcal{K}(\boldsymbol{g})(x) = K\boldsymbol{g}(x), \quad \forall x \in \Omega.$$

where $K \in \mathbb{R}^{m \times m}$ and $K_{ij} = \alpha_{ij}$; x^+ is the state after one-step.

Introduction to Koopman Operator

$$\boldsymbol{g}(x^+) = \boldsymbol{g}^+(x) := \mathcal{K}(\boldsymbol{g})(x) = K\boldsymbol{g}(x), \quad \forall x \in \Omega.$$

We say $\{\boldsymbol{g}, K\}$ a finite-dimensional representation of f.

- K represent a linear relationship in $\mathcal{H}$.
- K connects the elements in the function space instead of finite-dimensional space like $\mathbb{R}^n$.
- A linear system that directly captures measurement evolution and that indirectly monitors state evolution.

Idea of Koopman Operator

Transform a nonlinear system in finite-dimensional spaces into a linear system in infinite-dimensional spaces.

Introduction to Koopman Operator

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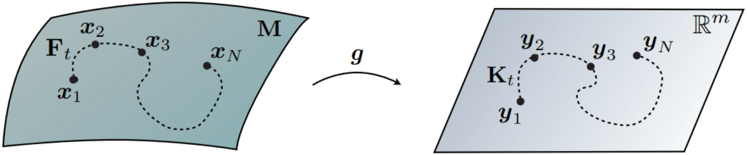

Figure: Illustration of state trajectories x_t and observable trajectories $y_t := \boldsymbol{g}(x_t)$ (Brunton and Kutz, 2019).

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Koopman Invariant Subspace

Definition

A subspace $\mathcal{M} \subset \mathcal{H}$ is a Koopman invariant subspace if $\mathcal{K}(g) \in \mathcal{M}$ $\forall g \in \mathcal{M}$.

If $\mathcal{M}$ is spanned by finite functions $\{g_1, \ldots, g_m\}$, i.e., for any $g \in \mathcal{M}$, there exists $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$ such that

$$g = \alpha_1 g_1 + \cdots + \alpha_m g_m, \quad \text{and} \quad \mathcal{K}(g) = \beta_1 g_1 + \cdots + \beta_m g_m.$$

- $\{g_1, \ldots, g_m\}$ are bases of the invariant subspace.
- $\mathcal{K}$ has a finite-dimensional representation $K \in \mathbb{R}^{m \times m}$ on $\mathcal{M}$.
- $\boldsymbol{\beta} = K\boldsymbol{\alpha}$.

Koopman Eigenfunction

$\mathcal{K}$ is a linear operator, we can define eigenvalue and eigenfunctions.

Definition

λ and $\phi_\lambda(x)$ are the eigenvalue and eigenfunction of $\mathcal{K}$ if

$$\mathcal{K}(\phi_\lambda)(x) = \lambda\phi_\lambda(x)$$

- In general, $\lambda \in \mathbb{C}$.
- Any finite eigenfunctions form an invariant subspace.
- If $\{g_1, \ldots, g_m\}$ spans a Koopman invariant subspace, we can find m eigenfunctions that spans the same subspace.

Example — Observables as Coordinates

Consider the dynamical system with $x \in \mathbb{R}^2$:

$$x_{1,t+1} = ax_{1,t}, \quad x_{2,t+1} = bx_{2,t} + (b - a^2)x_{1,t}^2.$$

We define three observables by $g_1(x) = x_1$, $g_2(x) = x_2$, $g_3(x) = x_1^2$. Then $\mathcal{K}(\mathbf{g})(x_t)$ can be represented by

$$\begin{bmatrix} g_1(x_{t+1}) \\ g_2(x_{t+1}) \\ g_3(x_{t+1}) \end{bmatrix} = \mathcal{K}(\mathbf{g})(x_t) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & b - a^2 \\ 0 & 0 & a^2 \end{bmatrix} \begin{bmatrix} g_1(x_t) \\ g_2(x_t) \\ g_3(x_t) \end{bmatrix}.$$

Let $y_t = \mathbf{g}(x_t)$, we have $y_{t+1} = Ky_t$.

Example — Eigenfunctions as Coordinates

Consider the dynamical system

$$x_{1,t+1} = ax_{1,t}, \quad x_{2,t+1} = bx_{2,t} + (b - a^2)x_{1,t}^2.$$

We define three observables by $\phi_1(x) = x_1, \phi_2(x) = x_2 + x_1^2, \phi_3(x) = x_1^2$.

$$\begin{bmatrix} \phi_1(x_{t+1}) \\ \phi_2(x_{t+1}) \\ \phi_3(x_{t+1}) \end{bmatrix} = \mathcal{K}(\phi)(x_t) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^2 \end{bmatrix} \begin{bmatrix} \phi_1(x_t) \\ \phi_2(x_t) \\ \phi_3(x_t) \end{bmatrix}.$$

Let $z_t = \phi(x_t)$, we have $z_{t+1} = \Lambda z_t$.

- $\{g_1, g_2, g_3\}$ and $\{\phi_1, \phi_2, \phi_3\}$ span the same invariant subspace.
- K is not diagonal while Λ is diagonal.

Koopman Mode Decomposition

Suppose $\boldsymbol{g}$ and ϕ span the same invariant subspace. K and Λ are their finite-dimensional representations.

Definition

We can decompose any function g_i into the eigenfunction bases.

$$g_i = \langle g_i, \phi_1 \rangle \, \phi_1 + \cdots + \langle g_1, \phi_m \rangle \, \phi_m.$$

The coefficients $v_{ij} = \langle g_i, \phi_j \rangle$ are the Koopman modes corresponding to the eigenfunction ϕ_j, $j = 1, \ldots, m$.

- Koopman modes measure the impact of g_i in the direction of ϕ_j.
- K and Λ share the same eigenvalues. $K = V \Lambda V^{-1}$.
- The i-th row of V provides the Koopman modes to decompose g_i.

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Short Summary

Koopman operator theory

- perceives the state evolution indirectly through $\boldsymbol{g}$ and $\mathcal{K}$;
  - An indirect approach for system identification.
  - Estimations on $\boldsymbol{g}$ and $\mathcal{K}$ are sufficient; no need for f.
- provides a linear relationship between elements in function spaces.

Challenges to use Koopman operator:

- Find the invariant space and the finite-dimensional representation.
  - Choose the right bases $\boldsymbol{g}$ (or eigenfunctions) of the invariant space.
- Estimate $\mathcal{K}$ from observation data.
- Restore real states from observations if necessary.

Method Overview

In general, $\boldsymbol{g} = \{g_1, \ldots, g_m\}$ may not be an invariant subspace.

We approximate of finite-dimensional representation:

$$\min_K \quad \|\boldsymbol{g} \circ f - K\boldsymbol{g}\| := \int_\Omega |\boldsymbol{g} \circ f(x) - Kg(x)| \, dx.$$

- Function norm.
- Complex or unknown f. Data-driven methods.

Two categories for data-driven methods:

- Choose $\boldsymbol{g}$ and learn K: DMD, Extended DMD, Hankel DMD.
- Learn $\boldsymbol{g}$ and K: Deep Koopman.

Dynamic Mode Decomposition

We have N trajectory data

$$X = [x_1, x_2, \ldots, x_N], \quad X' = [x'_1, x'_2, \ldots, x'_N]$$

with $x'_n = f(x_n)$. We choose the measurement $\boldsymbol{g} : \Omega \to \mathbb{R}^m$:

$$Y = \boldsymbol{g}(X) = [\boldsymbol{g}(x_1), \boldsymbol{g}(x_2), \ldots, \boldsymbol{g}(x_N)], \quad Y' = \boldsymbol{g}(X').$$

Least square estimation:

$$\min_K \left\| Y' - KY \right\|_2^2 = \sum_{i=1}^{N} \left\| \boldsymbol{g}(x'_i) - K\boldsymbol{g}(x_i) \right\|_2^2 .$$

- Optimal solution $K^* = Y'Y^\dagger$.

Data-Driven Approaches

Dynamic Mode Decomposition (DMD) and its variants:

- DMD (Tu et al., 2014).
  - full observation: $g_i(x) = x_i$ and $Y = X$.
- Extended DMD (Williams et al., 2015).
  - Choose nonlinear $\boldsymbol{g}$.
- Hankel DMD (Arbabi and Mezic, 2017).
  - Use delay-embedding of measurements on the observables.
- Sparse identification of nonlinear dynamics (Brunton et al., 2016).

Finding Koopman invariant subspace:

- Learning invariant subspace bases (Takeishi et al., 2017).
- Learning eigen-functions (Lusch et al., 2018), K is diagnoal.
- Learning both K and bases $\boldsymbol{g}$ (Yeung et al., 2019).

Related Literature

Koopman operator

- Begin with the seminal works (Koopman, 1931; Koopman and Neumann, 1932).
- Gains the renaissance from 1990s (Mezic, 1994; Mezić, 2005; Rowley et al., 2009).
- Review on Koopman operator (Brunton et al., 2022; Bevanda et al., 2021; Mezić, 2021).
- Survey on vehicular applications using Koopman operator (Manzoor et al., 2023).

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Koopman Operator Spectral Decomposition

Is the following correct?

$$\mathcal{K}(g)(x) = \sum_{\lambda \in \sigma(\mathcal{K})} \lambda v_\lambda \phi_\lambda(x), \quad \forall g \in \mathcal{H},$$

where $\sigma(\mathcal{K})$ is the set of all eigenvalues and v_λ is the Koopman mode.

Koopman Operator Spectral Decomposition

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where $\sigma(\mathcal{K})$ is the set of all eigenvalues and v_λ is the Koopman mode.

The answer is NO.

Because the Koopman operator $\mathcal{K}$ has both discrete and continuous spectrum (Mezić, 2005; Colbrook and Townsend, 2024).

Koopman Operator Spectral Decomposition

$\mathcal{K}$ is unitary, its spectrum is inside the unit circle $\mathbb{T}$ in the complex plane. We decompose it into singular (discrete) and regular (continuous) parts[2]:

$$\mathcal{K} = \mathcal{K}_s + \mathcal{K}_r = \sum_{\lambda \in \sigma_s(\mathcal{K})} \lambda \mathcal{P}_\lambda + \int_{\mathbb{T} \setminus \sigma_s(\mathcal{K})} y d\mathcal{E}(y).$$

- $\mathcal{P}_\lambda : \mathcal{H} \to \mathcal{H}$ is the projection operator to the eigenspace associated with the eigenvalue λ. $\mathcal{P}_\lambda(g) = \frac{\langle g, \phi_\lambda \rangle}{\langle \phi_\lambda, \phi_\lambda \rangle} \phi_\lambda$, $g \in \mathcal{H}$.
- $\mathcal{E}$ is the continuous spectral measure (eigenmeasure) that is "continuously parameterized" by y.
- $\mathcal{E}(U) : \mathcal{H} \to \mathcal{H}$ is the spectral projector to the subspace spanned by the "continuous eigenfunctions" with eigenvalues in U, U is any Borel measurable subset of $\mathbb{T}$.

[2] Lebesgue measure decomposition.

Koopman Operator Spectral Decomposition

Koopman mode decomposition for an arbitrary $g \in \mathcal{H}$:

$$\mathcal{K}(g)(x) = \sum_{\lambda \in \sigma_s(\mathcal{K})} \lambda v_\lambda \phi_\lambda(x) + \int_{\mathbb{T} \setminus \sigma_s(\mathcal{K})} y \psi_{g,y}(x) dy.$$

- v_λ is expansion coefficient $\frac{\langle g, \phi_\lambda \rangle}{\langle \phi_\lambda, \phi_\lambda \rangle}$. Koopman mode.
- $\psi_{g,y}$ is a "continuously parameterized" collection of eigenfunctions. we can understand it as $d\mathcal{E}(y)g$.
- Change of variable $y = e^{i\theta}$ and convert $\mathbb{T}$ to $[-\pi, \pi]_{\mathrm{per}}$.

We arrive at

$$g(x_t) = \mathcal{K}^t(g)(x_0) = \sum_{\lambda \in \sigma_s(\mathcal{K})} v_\lambda \lambda^t \phi_\lambda(x_0) + \int_{[-\pi,\pi]_{\mathrm{per}}} e^{it\theta} \psi_{g,\theta}(x_0) d\theta.$$

Koopman Operator Spectral Decomposition

Why spectral decomposition is useful?

Check if $g \in \mathcal{H}$ has discrete/continuous part.

$$d\nu_g(z) = \sum_{\lambda \in \sigma_s(\mathcal{K})} \langle \mathcal{P}_\lambda(g), g \rangle \, \delta(z - \lambda) dy + \int_{\mathbb{T} \setminus \sigma_s(\mathcal{K})} \psi_{g,y}(z) dy$$

with $g(z) = \int_\Omega d\nu_g(z) dz$.

Example Revisited

$$x_{1,t+1} = a x_{1,t}, \quad x_{2,t+1} = b x_{2,t} + (b - a^2) x_{1,t}^2.$$

We choose $g_1(x) = x_1$, $g_2(x) = x_2$, $g_3(x) = x_1^2$. $\boldsymbol{g}$ does not have continuous spectrum when projected to the eigenspace of $\mathcal{K}$.

Koopman Operator Spectral Decomposition

Generally, for a $g \in \mathcal{H}$,

- discrete part is dominant;
- continuous part is related to chaotic components of f.

Choosing proper $\boldsymbol{g}$ is important. Generally, see (Colbrook and Townsend, 2024)

- smooth g will have continuous part but easy to compute;
- nonsmooth g have less continuous part but hard to compute.

Finding (λ, ϕ_λ) is important.

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Spectrum Computation

A procedure related to ResDMD (Colbrook and Townsend, 2024; Colbrook et al., 2023):

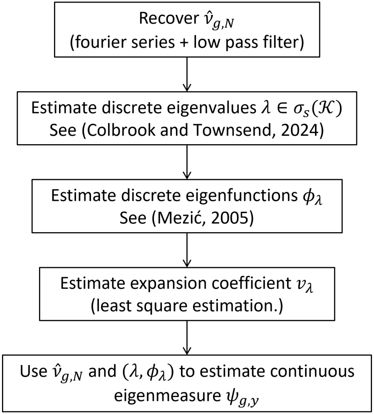

Figure: Procedures to recover eigenvalues and eigenfunctions of $\mathcal{K}$.

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Extending Koopman Operator for Control

Koopman operator

- Enables data-driven methods for indirect system identification.
- Works for autonomous dynamical systems.

We are interested in

- Data-driven methods for control.
- Applying Koopman operator to control.

Extending Koopman Operator for Control

Given a dynamical control system

$$x_{t+1} = f(x_t, u_t), \tag{3}$$

where $x_t \in \Omega \subset \mathbb{R}^n$, $u_t \in \mathcal{U} \subset \mathbb{R}^m$, $f : \Omega \times \mathcal{U} \to \Omega$.

Basic idea:

- Reflect the evolution of f and the impact of an arbitrary u.
- Extend the state space to $\Omega \times \mathcal{U}$.

We define the Koopman operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$

$$\mathcal{K}(g)(x_t, u_t) = g(f(x_t, u_t), u_{t+1}) = g(x_{t+1}, u_{t+1}), \tag{4}$$

where $g : \Omega \times \mathcal{U} \to \mathbb{R}$ is a observable in $\mathcal{H}$.

Control Variants

Different forms of control:

- Closed loop control: $u_t = h(x_t)$.

$$\mathcal{K}(g)(x_t, h(x_t)) = g(x_{t+1}, h(x_{t+1})).$$

  Reduce to Koopman operator for the associated autonomous system.

- Open loop control with internal control dynamics: $u_{t+1} = h(u_t)$.

$$\mathcal{K}(g)(x_t, u_t) = g(f(x_t, u_t), h(u_t)).$$

  Reduce to Koopman operator for the associated autonomous system where u is also a state.

- Open loop control with exogenous controls: unknown inputs.

Observable Bases

Next step: find Koopman invariant subspace and linearization.

We select $\boldsymbol{g} = [g_1, \dots, g_p]$ such that $\mathcal{K}(\boldsymbol{g}) \in \text{span}\{g_1, \dots, g_p\}$. i.e.,

$$\boldsymbol{g}(x_{t+1}, u_{t+1}) \approx K\boldsymbol{g}(x_t, u_t).$$

- Eigen-functions are also viable choices.

$$\mathcal{K}\phi_i(x, u) = \lambda_i \phi_i(x, u), \quad i = 1, 2, \dots.$$

Observable Bases — Special Structures

People assume special structures on the observable g for control.

- Partition g into two parts:

$$g(x,u) = g_x(x,u) + g_u(x,u).$$

- First part is only related to the state:

$$g_x(x,u) = g_x(x).$$

- Linear[3] of bilinear structure in the second part:

$$g_u(x,u) = a^\mathsf{T} u, \quad \text{or} \quad g_u(x,u) = \psi(x)(a^\mathsf{T} u).$$

[3] Linear structure is the most used case.

Observable Bases — Special Structures

Using linearity and causality (Korda and Mezić, 2018), we can write

$$\boldsymbol{g}(x, u) = \begin{bmatrix} \boldsymbol{g}_x(x) & u \end{bmatrix}^\mathsf{T}.$$

Then we have

$$\boldsymbol{g}(x_{t+1}, u_{t+1}) = \begin{bmatrix} \boldsymbol{g}_x(x_{t+1}) \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} K_{xx} & K_{xu} \\ K_{ux} & K_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_x(x_t) \\ u_t \end{bmatrix}. \tag{5}$$

We get rid of u_{k+1} since we do not predict controls, resulting in

$$\boldsymbol{g}_x(x_{t+1}) = K_{xx}\boldsymbol{g}_x(x_t) + K_{xu}u_t. \tag{6}$$

Example — System with Control

Consider the dynamical system with control:

$$x_{1,t+1} = \mu x_{1,t}, \quad x_{2,t+1} = \lambda(x_{2,t} - x_{1,t}^2) + \delta u_t.$$

We define $g_1(x,u) = x_1$, $g_2(x,u) = x_2$, $g_3(x,u) = x_1^2$, $g_4(x,u) = u$. Then $\boldsymbol{g}_x(x) = [g_1(x)\ g_2(x)\ g_3(x)]$. $\mathcal{K}(\boldsymbol{g})(x_t, u_t)$ can be represented by

$$\boldsymbol{g}_x(x_{t+1}) = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \mu^2 \end{bmatrix} \boldsymbol{g}_x(x_t) + \begin{bmatrix} 0 \\ \delta \\ 0 \end{bmatrix} g_4(u_t).$$

Let $y_t = \boldsymbol{g}_x(x_t)$, we have $y_{t+1} = Ay_t + Bu_t$.

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Transform Nonlinear Optimal Control Problem

Nonlinear optimal control problem (NOCP):

$$\begin{aligned} \min \quad & l_T(x_T) + \sum_{t=0}^{T-1} l_t(x_t) + u_t^\top R_t u_t + r_t^\top u_t \\ \text{s.t.} \quad & x_{t+1} = f(x_t, u_t), \quad t = 0, \dots, T-1, \\ & h_t(x_t) + c^\top u_t \le 0, \quad t = 0, \dots, T-1, \\ & h_T(x_T) \le 0. \end{aligned} \tag{7}$$

Tricks to select $\boldsymbol{g}_x$:

- Augment state itself: $\boldsymbol{g}_x = [x, \tilde{g}]$. ($C = [I \;\; 0]$, $x = Cg_x$).
- Augment nonlinear functions in the NOCP: $\boldsymbol{g}_x = [\tilde{g}, l_0, \dots, l_T, h_0, \dots, h_T]$.

Transform Nonlinear Optimal Control Problem

- Let $z_t = \boldsymbol{g}_x(x_t)$.
- Compute finite-dimensional Koopman operator K.
- Find A and B for dynamical systems.
- Convert nonlinear constraints.

Linearized optimal control problem:

$$\begin{aligned} \min \quad & y_T^\top Q_T y_T + \sum_{t=0}^{T-1} y_t^\top Q_t y_t + u_t^\top R_t u_t + r_t^\top u_t \\ \text{s.t.} \quad & y_{t+1} = A y_t + B u_t, \quad t = 0, \dots, T-1, \\ & E_t z_t + F_t u_t \leq 0, \quad t = 0, \dots, T-1, \\ & z_0 = \boldsymbol{g}_x(x_0). \end{aligned} \tag{8}$$

Discussions

Questions:

- Why do we partition $\boldsymbol{g}$ into two parts?
- Why do we assume linear or affine structure in u rather than in x?
- Why do we need linear-quadratic structure in u in NOCP (7)?

Discussions:

- Partition provides a notion of "control" in the lifted linear system. More convenient to process.
- x can be unknown but we must know u. Otherwise, we cannot control the original system.
- Linear or affine structure allows us access u directly. Otherwise, we need to learn the inverse function $\boldsymbol{g}_u^{-1}$ to perform control.
- Linear-quadratic structure in u is required by the linear structure in $\boldsymbol{g}$.

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Date-Driven Approaches

We have N trajectory data:

$$X = [x_1, x_2, \ldots, x_N],\ U = [u_1, u_2, \ldots, u_N],\ X' = [x'_1, x'_2, \ldots, x'_N].$$

with $x'_i = f(x_i, u_i)$. Linear structure in control: $\boldsymbol{g} = [\boldsymbol{g}_x \ \ I]$.

Least square estimation:

$$\min_{K_x, K_u} \quad \left\| K_x \boldsymbol{g}_x(X) + K_u U - \boldsymbol{g}_x(X') \right\|_2^2 .$$

- Extended DMD, the bases $\boldsymbol{g}$ are given (Korda and Mezić, 2018).
- Deep learning on $\boldsymbol{g}$ and K (Shi and Meng, 2022).
  - K step prediction loss.
  - Add regularization if necessary.

Related Literature

Koopman operator for control

- Starts from Korda and Mezić (2018); Proctor et al. (2018).
- Widely used in many fields, including robotics, aerospace, and traffic. See Manzoor et al. (2023).

Other approaches to system identification for control.

- Dynamic Mode Decomposition with control (DMDc) (Proctor et al., 2016).
- SINDy for model predictive control (Kaiser et al., 2018).
- Neural networks for model predictive control (Chen et al., 2018; Li et al., 2019; Drgona et al., 2020).

Related Literature

Koopman control in robotics[4]

- Soft robots (Bruder et al., 2019, 2020; Wang et al., 2022; Alora et al., 2023).
- Rehabilitation (Goyal et al., 2022).
- Human-robot interaction (Broad et al., 2020)
- UAV/UGV (Folkestad et al., 2020; Ren et al., 2022).
- Manipulator (Zhang and Wang, 2023).
- General learning for control systems and applications in robotics
    - Deep learning (Shi and Meng, 2022; Yin et al., 2022).
    - Bilinear Koopman operator (Bruder et al., 2021).
    - Stable koopman operator (Mamakoukas et al., 2023).
    - Control affine systems (Guo et al., 2021).
    - Derivative-based Koopman operator and error bound (Mamakoukas et al., 2021).
    - ...

[4] Soft robots are the most studied application area. General frameworks for Koopman learning are widely discussed.

Recommended Reference

Koopman operator theory

- Brunton et al. (2022); Bevanda et al. (2021); Mezić (2021).

Learning methods on Koopman operator

- Takeishi et al. (2017); Lusch et al. (2018); Yeung et al. (2019).

Spectral Properties of Koopman operator

- Mezić (2005); Colbrook and Townsend (2024).

Koopman operator for control

- Korda and Mezić (2018); Proctor et al. (2018); Shi and Meng (2022).

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